DISSERTATION REVIEW REPORT

Title of dissertation: NEW ALGORITHMS FOR BERNSTEIN POLYNOMIALS, THEIR DUAL BASES, AND B-SPLINE FUNCTIONS

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1. Research description and most significant contributions

The main topic of this thesis is the development of very efficient (or even optimal) algorithms to evaluate and work with (i) Bézier curves, (ii) B-spline curves, and (iii) dual Bernstein polynomials.

(i) Regarding Bézier curves, the main result consists in the proposal of a new, optimal algorithm for their evaluation, which is the only one to melt together the low (linear) complexity of the Horner's scheme with the geometric feature of de Casteljau's algorithm, well-known in geometric design for the advantage of relying only on convex combinations.
The new evaluation algorithm for polynomial Bézier curves is also generalized to rational Bézier curves and, more generally, to rational parametric objects such as rational rectangular as well as triangular Bézier surfaces. Even in such more general cases, the algorithm has optimal complexity, i.e., it is proportional to the number of control points that define the objects.
Among the multiple applications of this really important evaluation algorithm, it is shown how to accelerate the evaluation of many polynomial B-spline curves at multiple points, thus reducing the complexity of the well-known de Boor-Cox algorithm.

(ii) Regarding polynomial B-splines, the main result deals with the introduction of a new differential-recurrence relation satisfied by B-spline functions, and on the consequent derivation of a new algorithm for computing the Bernstein-Bézier coefficients of B-spline functions. The resulting algorithm turns out to be optimal if the knots of the B-spline functions satisfy some assumptions. Moreover, if the Bernstein-Bézier coefficients of the B-spline functions are known, one can perform the evaluation of each function in linear time with respect to its coefficients.

(iii) Finally, regarding dual Bernstein polynomials, several new diferential, differential-recurrence, and recurrence relations have been introduced. Thanks to such relations it is possible to solve several problems that range from the computation of a linear combination of dual Bernstein polynomials of degree n in the optimal O(n) time, to the parallel constrained degree reduction of a Bézier curve.

These results are collected in five chapters (numbered from 2 to 6). The chapter that precedes these ones (i.e., Chapter 1) is an introductory chapter devoted to recall the needed background and acquaint the reader with the notation used in the following chapters.

Chapter 2 starts with the results already published by Filip Chudy in

P. Wozny and F. Chudy. Linear-time geometric algorithm for evaluating Bézier curves. *Computer Aided-Design*, 118:102760, 2020.

In this paper, the focus is on Bézier curves. However, the proposed algorithm is not limited to curves, but general enough to be applied to many other (rational) parametric objects, provided that their control points and basis functions can be ordered in such a way that the algorithm's principles are satisfied. The second part of Chapter 2 indeed extends the results previously published in CAD 2020 from rational Bézier curves to rational rectangular Bézier surfaces and rational triangular Bézier surfaces.

The content of Chapters 4 and 5 is strongly related to the results published by Filip Chudy in

F. Chudy and P. Wozny. Differential-recurrence properties of dual Bernstein polynomials. *Applied Mathematics and Computation*, 338:537–543, 2018.

F. Chudy and P. Wozny. Fast and accurate evaluation of dual Bernstein polynomials. *Numerical Algorithms*, 87:1001–1015, 2021.

The differential-recurrence relations shown to be satisfied by dual Bernstein polynomials are used to derive differential equations for dual Bernstein polynomials (Chapter 4), and to find a recurrence relation of order 4 for dual Bernstein polynomials of the same degree (Chapter 5). In Chapter 5 new recurrence relations of lower order are also presented together with algorithms for evaluating dual Bernstein polynomials. The numerical results show that the proposed evaluation algorithm works very well even for high degrees of dual Bernstein polynomials.

Differently from Chapters 2, 4 and 5, the content of Chapters 3 and 6 has not yet been published. However, one could extract from such chapters also some publishable material.

Indeed, Chapter 3 exploits a new differential-recurrence relation for B-spline functions over multiple knot spans to derive computationally simple recurrence relations for the coefficients of the adjusted Bernstein-Bézier form. When such coefficients are known, one can use the new algorithm given in Chapter 2 to evaluate a B-spline function of degree m in O(m) time or a B-spline curve, which, when evaluating many curves at multiple points, has lower computational complexity than using the de Boor-Cox algorithm. Thus, by accompanying such results with some examples of practical contexts where it is crucial to have multiple B-spline curve evaluations, the content of Chapter 3 could have the real potential to become publishable.

Regarding Chapter 6, it is here provided an application of the new recurrence relations given in Chapter 5 in the context of degree-reduction of Bézier curves. Precisely, in Chapter 6, it is studied how to compute, from a Bézier curve of degree n, a Bézier curve of degree $m<n$ which satisfies some constraints on the derivatives at the end points of the domain [0,1] (which are aimed at preserving the shape of the curve at its ends), and minimizes the value of the integral that identifies its optimality in the sense of the least-square approximation related to the shifted Jacobi scalar product.
Finding this Bézier curve of reduced degree requires computing O(nm) scalar products of Bernstein polynomials and their dual counterparts of different degrees. These products (suitably scaled by a certain factor) are arranged into a table. The evaluation of the elements of such a table relies on the previously proposed recurrence relations. Since the method given by Wozny and Lewanowicz in [97] has the same O(nm) complexity, there is no improvement in this respect. However, the method proposed in Chapter 6 offers the possibility of parallelizing a significant part of the computations. Thus, by adequately emphasizing the application contexts that could benefit from the parallelized version of the algorithm, the content of Chapter 6 could certainly become publishable.

Looking at the candidate's cv it turns out that he has already planned the submission of the following three articles:

(a) F. Chudy and P. Wozny, Fast evaluation of Bernstein-Bézier and power basis coefficients of B-spline functions.

(b) F. Chudy and P. Wozny, Fast parallel (k,l)-constrained Bézier curve degree reduction.

(c) F. Chudy and P. Wozny, Efficient evaluation of Bézier-type objects and their derivatives.

While (a) and (b) are related to the content of Chapter 3 and 6, (c) reveals that the author has in mind to publish also the unpublished content of Chapter 2, which extends the results previously published in CAD 2020. I think that there is indeed a good chance of success taking into account that the work appeared in CAD 2020 (which is restricted to the curve case) contains a very nice result, which I appreciated so much that I have already used for my personal research. I agree with the author that the extension to rational Bézier surfaces (both rectangular and even better triangular) is worthy of publication too.

2. Overall evaluation and recommendation

The reading of the manuscript was really pleasant since the dissertation is very well organized, self-contained, clearly written and nicely structured, so that it is easy for the reader to identify the main ideas and the original contributions. All the topics are presented in a precise and complete manner, giving to the reader the needed background. The bibliography is adequate, the problems to be treated and the previous known results correctly introduced. The new mathematical results are cleanly and shortly proven, the proposed innovative algorithms described in a meticulous and comprehensible manner, and the comparative studies conducted rigorously. I have found the use of schemes and tables to clarify the key features of the proposed algorithms and compare their accuracy, complexity and running times with known algorithms, extremely interesting and useful.

The criteria of originality and innovation are well-met and testified by the significant publications that have already arisen from the research activity of the candidate in the last few years. It is certainly not so common to arrive at the conclusion of the PhD thesis with three articles already published in high-ranked international journals like Computer Aided-Design, Numerical Algorithms, Applied Mathematics and Computation, as well as three additional articles in preparation, with a real potential of being published.

In summary, it is my opinion that this doctoral dissertation fully satisfies all the requirements to be judged an EXCELLENT doctoral thesis to be accepted unconditionally (i.e., without reserve).

I'm also in favour to nominate this dissertation for receiving a distinction and, if the Scientific Council considers the dissertation eligible for a scientific prize, I will be pleased to support this decision.

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